The Delsarte Method in the Problem of the Antipodal Contact Numbers of Euclidean Spaces of High Dimensions*

D. V. Shtrom

In this paper we study the Delsarte problem for even functions continuous on $[-1,1]$, nonpositive on $[-1/2,1/2]$, and representable as series with respect to the ultraspherical polynomials $\{R_n^{\alpha,\alpha}\}_{n=0}^{\infty}$, $\alpha=(m-3)/2$, $m\geq 2$, with nonnegative coefficients. The value w_m^A of the Delsarte problem gives an upper bound for the largest power of antipodal spherical 1/2-code of the space $\mathbb{R}^m$, $m\geq 2$. The Delsarte problem is the problem of infinite linear programming. In this paper the value w_m^A is found for $3\leq m\leq 161$ with a few gaps. The results are summarized in a table. We give a detailed proof for $m=43$ and point out a scheme of a proof for other cases.

1 Introduction. Setting of the problem

In this paper we study an extremal problem on a class of functions continuous on a closed interval, representable by series with respect to the ultraspherical polynomials with restrictions on values of the functions and coefficients of the representations. The problem occurs in applying of the Delsarte scheme to the problem on the largest power of antipodal spherical s-codes of Euclidean spaces $\mathbb{R}^m$.

For the first time this scheme arose in the investigations of Delsarte [1], [2] of the bounds of packings in certain metric spaces. The Delsarte scheme was developed and successfully used in the works of G.A. Kabatianskii and V.I. Levenshtein [3], A. Odlyzko and N. Sloane [4], V.I. Levenshtein [5], [6], [7], V.M. Sidel'nikov [8], V.A.Yudin [9], P.G. Boyvalenkov [10], P.G. Boyvalenkov, D.P. Danev, and S.P. Bumova [11], V.V.Arestov and A.G. Babenko [12], [13], and others in connection with the investigation of

*This work was supported by the Russian Foundation for Basic Research, project no. 02–01–00783, and by the Program for State Support of Leading Scientific Schools of Russian Federation, project no. 1347.2003.1.

optimal (in one or another sense) arrangement of points in metric spaces and, in particular, the investigation of the contact numbers of Euclidean spaces $\mathbb{R}^m$. The Delsarte scheme leads to a problem of infinite linear programming (we will call it the Delsarte problem). The value of the problem gives an upper bound for the original problem. Our interest is the Delsarte problem connected with spherical s-codes in $\mathbb{R}^m$. First in this theme, V.M.Sidel'nikov used in his work a quadrature formula of the Gauss–Markov type where only values of a function were equipped. He applied the formula to prove an extremality of the Levenshtein polynomials on a class of polynomials of the same or a smaller degree. In the paper [11] by P.G.Boyvalenkov, D.P.Danev, S.P.Bumova an extremality criterion for the polynomials of V.I.Levenshtein in the problem on the largest power of a spherical s-code was proved. In the paper [14] by P.G.Boyvalenkov and D.P.Danev, the criterion was generalized. This made possible the use of the criterion to establish values s and m for which the Levenshtein polynomials are the extremal polynomials in the problem on the largest power of an antipodal s-code (the antipodal contact number). Using this criterion, P.G.Boyvalenkov and D.P.Danev established the non-extremality of the Levenshtein polynomials for some s and m. But they failed to establish the extremality of the Levenshtein polynomials. V.V.Arestov and A.G.Babenko [12] constructed essentially a quadrature formula where not only values of functions, but also their Fourier–Jackobi coefficients were equipped. In this way they succeeded, in particular, in solving of the Delsarte problem for $s = 1/2, m = 4$. In the paper [15] the Delsarte problem was solved for $s = 1/3$, $m = 4$, 5, 6 by a similar method. The author [16] applied this method to find a solution of the Delsarte problem for $s = 1/2$ and the following m :

$$5 \leq m \leq 146,\ 148 \leq m \leq 156,\ m = 161,\ (m \neq 8, 24).$$

Let $\mathbb{R}^m, m \geq 2,$ be the real Euclidean space with the standard inner product $xy = x_1y_1 + x_2y_2 + \ldots + x_my_m,\ x = (x_1, x_2, \ldots, x_m),$ $y = (y_1, x_2, \ldots, y_m)$, and the norm $|x| = \sqrt{xx},\ x, y \in \mathbb{R}^m$. Let $\mathbf{B}^m(y) = \{x \in \mathbb{R}^m : |x - y| \leq 1\}$ be the ball of the unit radius with center at the point $y \in \mathbb{R}^m$. Let τ_m denote the maximal number τ of nonoverlapping balls $\mathbf{B}^m(y^{(1)}),\ \mathbf{B}^m(y^{(2)}),\ \ldots,\ \mathbf{B}^m(y^{(\tau)})$ of the unit radius: $(|y^{(i)}| = 2,$ $i = 1, 2, \ldots, \tau;\ \ |y^{(i)} - y^{(j)}| \geq 2,\ i \neq j)$ touching the central ball $\mathbf{B}^m(0)$ of the unit radius. The quantity τ_m is called the *contact number* of the space $\mathbb{R}^m$.

At present, the exact values of τ_m are known (see [17, Table 1.5 and Chapter 1]) only for $m = 2$, 3, 8, 24; namely $\tau_2 = 6$, $\tau_3 = 12$, $\tau_8 = 240$,

$\tau_{24} = 196560$. In the general case two-sided estimates of this value exist. Under additional restriction of symmetry with respect to origin on a location of balls, we obtain the problem of non-lesser interest about finding of *antipodal contact number* τ_m^A. This problem is connected (see, for instance, [9]) with the classical problem on the largest number of integer points on an ellipsoid. About these problems it is known a little bit more. In particular, it is known that $\tau_2^A = 6$, $\tau_3^A = 12$, $\tau_4^A = 24$, $\tau_5^A = 40$, $\tau_6^A = 72$, $\tau_7^A = 126$, $\tau_8^A = 240$, $\tau_{24}^A = 196560$. Moreover, 24 is the largest dimension where these values are calculated exactly. For $m > 24$ only two-sided estimates of these quantities are known. But the known estimates do not show, for instance, the order of growth neither of τ_m nor of τ_m^A as $m \to \infty$.

Specific arrangements of balls give lower bounds for τ_m^A and τ_m. A non-constructive method to estimate τ_m^A from below also exists. In this paper we will not discuss lower bounds for τ_m^A. The information and the bibliography concerning this theme can be found in the already cited monograph [17]. The Delsarte approach give an efficient upper bound for τ_m^A. We will state the method for a more general case for the problem on the largest power of spherical s-codes of Euclidean spaces $\mathbb{R}^m$.

Let $\mathbf{S}^{m-1} = \{x \in \mathbb{R}^m : |x| = 1\}$ be the unit sphere of the space $\mathbb{R}^m$. We basically follow the notation of [19]. Suppose a set $W \subset \mathbf{S}^{m-1}$ contains at least two points. All possible values of inner product of different vectors x, y from W form the set which we denote by $\mathcal{A}(W)$, i.e.,

$$\mathcal{A}(W) = \{xy : \quad x \neq y,\ x, y \in W\}.$$

The set $W \subset \mathbf{S}^{m-1}$ with the property $\mathcal{A}(W) \subset [-1, s]$ is called the *spherical s-code.* The set $W \subset \mathbf{S}^{m-1}$ is called the *centrosymmetrical set,* if for any point x from this set, the point $-x$ also belongs to the set. The set $W \subset \mathbf{S}^{m-1}$ which is the centrosymmetrical set and the spherical s-code simultaneously, is called *the antipodal spherical s-code.*

Let $\mathcal{W}_m^A(s)$ denote the set of all antipodal spherical s-codes from $\mathbf{S}^{m-1}$. Let $N_m^A = N_m^A(s)$ denote the largest possible power of antipodal spherical s-code, i.e., suppose that

$$N_m^A(s) = \max\{\mathrm{card}(W) : \ W \in \mathcal{W}_m^A(s)\}. \tag{1}$$

It is readily seen that $N_m^A(1/2)$ coincides with τ_m^A :

$$\tau_m^A = N_m^A(1/2), \quad m \geq 2. \tag{2}$$

A trivial connection between τ_m and τ_m^A follows from the fact that an antipodal spherical s-code is a spherical s-code. Namely,

$$\tau_m^A \leq \tau_m.$$

Let $R_k = R_k^{\alpha,\alpha}$, $k = 0, 1, 2, \dots$, be the system of ultraspherical polynomials (Gegenbauer polynomials) orthogonal on the closed interval $[-1, 1]$ with weight $(1 - t^2)^\alpha$, $\alpha = (m - 3)/2$, and normalized by the condition $R_k(1) = 1$. We denote by $\mathcal{F}_m^A = \mathcal{F}_m^A(s)$, $s \in [0, 1)$, the set of functions f continuous on $[-1, 1]$ with the following properties:

(1) the function f can be represented as the series

$$f(t) = \sum_{k=0}^{\infty} f_{2k} R_{2k}(t), \tag{3}$$

whose coefficients satisfy the conditions

$$f_0 > 0, \quad f_{2k} \geq 0,\ k = 1, 2, \dots, \quad f(1) = \sum_{k=0}^{\infty} f_{2k} < \infty; \tag{4}$$

(2) the function f is nonpositive on $[-s, s]$:

$$f(t) \leq 0, \quad t \in [-s, s]. \tag{5}$$

On this set of functions we consider the Delsarte problem of evaluating

$$w_m^A(s) = \inf \left\{ \frac{f(1) + f(-1)}{f_0} : \ f \in \mathcal{F}_m^A(\mathrm{s}) \right\} = \inf \left\{ \frac{2\, f(1)}{f_0} : \ f \in \mathcal{F}_m^A(\mathrm{s}) \right\}. \tag{6}$$

Let agree to call this quantity the Delsarte constant (function).

In the paper [12] by V.V.Arestov and A.G.Babenko, it was proved finiteness of the expansion (3) of the function extremal in the problem (6), i.e., it was proved that the function providing the minimum in the problem (6) is a polynomial. At the same time, using Corollary 2.2 from [12], pointwise estimates for the polynomials $R_{2k}(t)$ on $[-s, s]$ (one can use, for instance, Lemma 2.1 from [16]), and also an universal upper estimate by V.I.Levenshtein for $w_m^A(s)$, one can find an efficient upper estimate of an degree of an extremal polynomial.

The following statement is contained in the paper [9]. This statement gives an upper estimate of $N_m^A(s)$, and so of τ_m^A (see (2)), by (6).

Theorem A. *For any $s \in [0, 1)$, $m = 2, 3, \dots$ we have*

$$N_m^A(s) \leq w_m^A(s). \tag{7}$$

Since N_m^A is an even number, the theorem involves

$$N_m^A(s) \leq [w_m^A(s)]_2, \quad s \in [0, 1), \quad m = 2, 3, \dots , \tag{8}$$

where $[t]_2$ is the largest even number not exceeding t.

The problem on the largest power of antipodal spherical codes was considered in the papers [6], [7], [18], [10]. In the paper by V.I.Levenshtein a good upper estimate of w_m^A was obtained. Bellow in the present paper it will be shown that for rather large number of m this estimate can not be improved by the Delsarte method. In the papers [14] and [18] necessary and sufficient conditions for extremality of the Levenshtein polynomials in the Delsarte problem were established. In the paper [18] with use of these conditions (criterion), a number of s and m for which the Levenshtein polynomial is not the extremal polynomial in the Delsarte problem were found. With use of the fact, in the paper [18] the estimate by V.I.Levenshtein was a little bit improved for these s and m. But neither in this paper nor in more earlier papers extremality of the Levenshtein polynomials (corresponding sufficient conditions) was not under consideration with the exception of those cases when an upper estimate coincides with the known lower estimate. In these cases extremality of the Levenshtein polynomials is obtained automatically (for example, for $s = 1/2$ and $m = 2, 4, 6, 7, 8, 24$).

Our interest is $s = 1/2$. Let

$$w_m^A = w_m^A(1/2).$$

The following is done in this paper.

(1) Exact values of the Delsarte constant w_m^A are evaluated for many m; namely, for all natural $3 \le m \le 99$, $104 \le m \le 122$, $125 \le m \le 134$, $136 \le m \le 145$, $147 \le m \le 156$, $m = 161$ ($m \ne 4, 6, 7, 8, 24$). Corresponding extremal polynomials are found. Notice that polynomials of the forms 1 and 2 (see Sect. 4) are (to within a change of variable) the polynomials obtained in the paper by V.I.Levenshtein [7]. For all these m it was, practically, needed to verify sufficient conditions for extremality of the Levenshtein polynomials in the Delasarte problem. And it was done. The corresponding results are given in Table 1. In addition, extremal polynomials different from the Levenshtein polynomials were found. These are polynomials of the forms 3 and 4 (see Sect. 4). The corresponding results are given in Table 2.

(2) We give a detailed proof for $m = 43$. The other results are given in a table without detailed proof because the corresponding computations occupy too much room. But the table allows one to carry out the computations for any m indicated in it similarly to the case $m = 43$.

By now, a few s and m are known for which $w_m^A(s)$ is an integer and coincides with $N_m^A(s)$, for example, for $s = 1/2$ and m equal to $2, 4, 6, 7, 8, 24$. But whether the opposite is true? And $N_m^A(s)$ coincides at least with $[w_m^A(s)]_2$

whether or not? Using the results of the paper by P.G.Boyvalenkov [10] and this paper, one can respond to the question as follows: given quantities are not equal in general. For example, $\tau_5^A = N_5^A(1/2) = 40$ but $w_5^A = 42$. It is also known [10] that $\tau_{10}^A \le 548$, $\tau_{14}^A \le 2938$, while $w_{10}^A = 550$ and $w_{14}^A = 2940$. Thus, we have $\tau_m^A < [w_m^A]_2$ for $m = 5, 10, 14$.

2 A method of investigation for the Delsarte problem.

From now on we follow in general the scheme of reasoning from the papers [12], [13], [15], [16]. The present paper is close to the paper [16] with respect to methods of investigation and nature of results.

Suppose $\ell = \ell_1$ is the space of summable sequences $x = \{x_{2k}\}_{k=1}^{\infty}$ of real numbers, and $C[-1,1]$ is the space of functions continuous on $[-1,1]$. Let A the linear operator from $\ell = \ell_1$ to $C[-1,1]$ defined by

$$(Ax)(t) = \sum_{k=1}^{\infty} x_{2k} R_{2k}(t), \quad t \in [-1,1],\ x = \{x_{2k}\}_{k=1}^{\infty} \in \ell.$$

Suppose

$$u_m^A(s) = \inf \left\{ \sum_{k=1}^{\infty} x_{2k} :\ x_{2k} \ge 0;\ 1 + (Ax)(t) \le 0,\ t \in [-s,s] \right\}. \qquad (9)$$

In the problem (6) we can restrict our attention to functions $f \in \mathcal{F}_m^A$, with $f_0 = 1$. For such a function let $x = \{f_{2k}\}_{k=1}^{\infty}$; we have $f(t) = 1 + (Ax)(t)$ and $f(1) = 1 + \sum\limits_{k=1}^{\infty} f_{2k}$. Hence, we can deduce that (6) is connected with (9) by

$$w_m^A(s) = 2 + 2u_m^A(s). \qquad (10)$$

The problems (6),(9) are problems of infinite linear programming (see, for instance, the monograph [20]). In [12], with the help of these considerations the existence of solutions (extremal functions) of a problem more general than (6) and of the corresponding dual problem was proved.

We denote by Φ_m^A the set of even functions $f \in C[-1,1]$ representable as the series with respect to the ultraspherical polynomials $R_{2k} = R_{2k}^{\alpha,\alpha}$, $\alpha = (m-3)/2$, with an (absolutely) summable sequence of real coefficients:

$$\Phi_m^A = \Big\{ f \in C[-1,1] : \quad f(t) = \sum_{k=0}^{\infty} f_{2k} R_{2k}(t), \ \ \sum_{k=0}^{\infty} |f_{2k}| < \infty \Big\}. \qquad (11)$$

Evidently, $\mathcal{F}_m^A \subset \Phi_m^A$.

In what follows, we consider only the case $s = 1/2$. To solve the problem (6), a quadrature formula (specific for each m) on the class of functions $f \in \Phi_m^A$ was constructed. This formula contains not only the values of the function, but also the Fourier coefficients $f_{2\nu}$ of the representation $f(t) = \sum_{k=0}^{\infty} f_{2k} R_{2k}(t)$ as the series with respect to the polynomials R_{2k}. To be exact, the formula has the form

$$f_0 = \frac{1}{\vartheta(\alpha)} \int_{-1}^{1} f(t)(1-t^2)^\alpha dt = L(f) - \sum_{\nu \geq 1} L(R_{2\nu}) f_{2\nu}, \tag{12}$$

where

$$\vartheta(\alpha) = \int_{-1}^{1} (1-t^2)^\alpha dt, \quad \alpha = (m-3)/2,$$

and the functional L is given by

$$L(f) = \lambda(1) f(1) + \lambda\left(\frac{1}{2}\right) f\left(\frac{1}{2}\right) + \lambda(t_1) f(t_1) + \ldots + \lambda(t_{k(m)}) f(t_{k(m)}), \tag{13}$$

where the nodes $t_1, t_2, \ldots, t_{k(m)}$ are from $[0, 1/2)$, the coefficients $\lambda\left(\frac{1}{2}\right)$, $\lambda(t_1)$, $\lambda(t_2), \ldots, \lambda(t_{k(m)})$ and the values $L(R_{2\nu})$, $\nu \geq 1$, of the functional L are nonnegative for the polynomials $R_{2\nu}$. Simultaneously with the quadrature formula (12) we construct a polynomial $f^* \in \mathcal{F}_m^A$ such that the following equality holds:

$$f_0^* = \frac{1}{\vartheta(\alpha)} \int_{-1}^{1} f^*(t)(1-t^2)^\alpha dt = \lambda(1) f^*(1). \tag{14}$$

Under these conditions, the quantity $2/\lambda(1)$ is the value of the problem (6) (for $s = 1/2$), i.e., we have

$$w_m^A = \frac{2}{\lambda(1)}. \tag{15}$$

Indeed, by (12) and the nonnegativity of the coefficients of the quadrature formula, for any function $f \in \mathcal{F}_m^A$ we have

$$f_0 = L(f) - \sum_{\nu \geq 1} L(R_{2\nu}) f_{2\nu} \leq L(f) \leq \lambda(1) f(1).$$

Thus, for any function $f \in \mathcal{F}_m^A$ we have

$$\frac{f(1)}{f_0} \geq \frac{1}{\lambda(1)} \tag{16}$$

and so

$$w_m^A \geq \frac{2}{\lambda(1)}. \tag{17}$$

For the polynomial f^* inequality (16) turns into equality. Consequently, inequality (17) is in fact an equality, i.e., (15) holds. In addition, the function f^* has the property $w_m^A = 2\, f^*(1)/f_0^*$, i.e., this polynomial is a solution (an extremal function) of the problem (6). The functional L is defined by the measure which is a solution of the dual problem.

The condition (14) imposes rather severe restrictions on the function

$$f^*(t) = \sum_{k=0}^{n(m)} f_{2k}^* R_{2k}(t), \tag{18}$$

and on the quadrature formula (13). Namely, the following conditions must be satisfied:

(a) all nodes t_ν of the functional L (except $t = 1$) belong to $[0, 1/2]$ and they are zeros of the function f^*. Moreover, every zero from $[0, 1/2)$ is at least a double zero;

(b) for $k \geq 1$ the coefficients f_{2k}^* of the representation (18) of the function f^* are connected with the values $L(R_{2k})$ of the functional (13) for the polynomials R_{2k} by the relation $f_{2k}^*\, L(R_{2k}) = 0$;

(c) the sum of weights of the functional L is equal to 1, i.e.,

$$L(R_0) = \lambda(1) + \lambda\left(\frac{1}{2}\right) + \lambda(t_1) + \lambda(t_2) + \ldots + \lambda(t_{k(m)}) = 1.$$

Besides, in order to the functional L be nonnegative and the polynomial f^* belong to the class, the following conditions must be satisfied:

(d) the weights $\lambda(t_\nu)$ of the functional L are nonnegative and $\lambda(1) > 0$;

(e) for all polynomials R_{2k}, $k \geq 1$, the functional L is nonnegative: $L(R_{2k}) \geq 0$, $k \geq 1$;

(f) the polynomial f^* belongs to the class $\mathcal{F}_m^A$, i.e., its coefficients f_{2k}^*, $k \geq 1$, are nonnegative, $f_0^* > 0$, and the condition $f^*(t) \leq 0$, $t \in [-1/2, 1/2]$, holds.

In all cases when the problem (6) is solved exactly, $t = 1/2$ is a zero of the polynomial, moreover, its multiplicity is 1. In order to construct the polynomial f^*, it is important to have a priori information about the structure of the polynomial: the degree, the numbers of the vanishing coefficients from the representation (18), the number of (multiple) zeros of the polynomial on $(-1/2, 1/2)$, and whether or not the point $t = -1$ is a zero of the

polynomial. Information about the form of the extremal function must be cleared up separately. Most often we actually guess it, using some ideas, for example, a preliminary numerical experiment. If we have the above information, the extremal polynomial f^* and the functional L are constructed as follows.

The conditions (a)–(c) give the system of (nonlinear) equations with respect to the nodes $\{t_i\}$ of the functional (13) from $[0, 1/2]$ that are zeros of the function f^*, the coefficients $\{f^*_{2k}\}$ of the function (polynomial) (18), and the weights $\{\lambda(t_i)\}$ of the functional (13). Usually, a solution of the set of equations is not unique. We have to choose a solution that satisfies the above-listed conditions (d)–(f).

3 The forms of extremal functions.

In this paper, the value of $w^A_m = w^A_m(1/2)$ will be found for the following m : $3 \le m \le 99$, $104 \le m \le 122$, $125 \le m \le 134$, $136 \le m \le 145$, $147 \le m \le 156$, $m = 161$ $(m \ne 4, 6, 7, 8, 24)$. In all these cases, the extremal function is a polynomial. The structure of the polynomial essentially depends on m. The extremal polynomials are of one of the following 4 forms. In formulas below a_i denote (double) zeros t_i of an extremal polynomial (nodes of the functional L) belonging to $(0, 1/2)$. In these formulas, and in what follows, $s = 1/2$.

1. $f(t) = (t^2 - s^2) \cdot \prod\limits_{i=1}^{K} (t^2 - a_i^2)^2,$
2. $f(t) = (t^2 - s^2) \cdot \prod\limits_{i=1}^{K} (t^2 - a_i^2)^2 \cdot t^2,$
3. $f(t) = (t^2 - s^2) \cdot \prod\limits_{i=1}^{K} (t^2 - a_i^2)^2 \cdot (t^4 + q\,t^2 + r), \quad f_{4K+2} = f_{4K+4} = 0,$
4. $f(t) = (t^2 - s^2) \cdot \prod\limits_{i=1}^{K} (t^2 - a_i^2)^2 \cdot (t^4 + q\,t^2 + r) \cdot t^2, \quad f_{4K+4} = f_{4K+6} = 0,$

The results of the work are summarized in the Tables 1 and 2 below. The first column gives the space dimension, the third column gives the number K of double zeros of the extremal function on $(0, 1/2)$, the fourth column gives one of 4 forms (exactly, the number of the form) of the extremal function, and the fifth column gives the degree of the obtained polynomial. The second column contains the solution of the Delsarte problem $w^A_m = w^A_m(1/2)$ that we have found.

The form of the extremal function and the values K, m is the very infor-

mation according to that we form the simultaneous equations starting from the conditions (a)–(c) of Section 2. In all cases considered in the paper, the number of equations coincides with the number of variables. The equations are nonlinear. In this connection, the solution is not unique. We have to choose the solution that gives a function f^* and a functional L satisfying the conditions (d)–(f). We have done this for all the above-mentioned m. But we cannot give full proof for each case because the corresponding computations occupy too much room. A detailed proof will be given only for $m = 43$. Constructing of the simultaneous equations for other m, their analysis and proof of extremality of obtained solutions are realized analogously.

All analytic and numerical computations were made by using the Maple package of analytic computations.

4 The table of the results.

In the Table 1 the values w_m^A are given for the case of an extremal polynomial having the form 1 or 2 (i.e., the Levenshtein polynomial). The indicated values were known earlier as estimates of the largest power of an antipodal spherical 1/2-code (see [7, p.8]). In the present paper we assert that the mentioned values are a solution of the problem (6) (for $s = 1/2$).

Table 1

m	w_m^A	K	Form	Degree
4	24	0	2	4
5	42	0	2	4
6	72	0	2	4
7	126	0	2	4
8	240	0	2	4
9	$366\frac{12}{73}$	1	1	6
10	550	1	1	6
11	$820\frac{16}{23}$	1	1	6
12	$1228\frac{1}{2}$	1	1	6
13	$1867\frac{17}{19}$	1	1	6
14	2940	1	1	6
15	$4962\frac{6}{37}$	1	1	6
16	8160	1	2	8
17	$11478\frac{12}{13}$	1	2	8

Table 1: (continued)

m	w_m	K	Form	Degree
18	$16122\frac{6}{7}$	1	2	8
19	22724	1	2	8
20	32340	1	2	8
21	$46879\frac{7}{17}$	1	2	8
22	$70165\frac{1}{3}$	1	2	8
23	$111126\frac{6}{19}$	1	2	8
24	196560	1	2	8
25	$267628\frac{622}{1451}$	2	1	10
26	364182	2	1	10
27	$497035\frac{7}{281}$	2	1	10
28	683240	2	1	10
29	$951235\frac{31}{251}$	2	1	10
30	$1352089\frac{1}{71}$	2	1	10
31	$1987341\frac{15}{197}$	2	1	10
32	$3091334\frac{2}{5}$	2	1	10
34	7314012	2	2	12
35	$9768755\frac{475}{733}$	2	2	12
36	13090896	2	2	12
37	$17663588\frac{56}{135}$	2	2	12
38	$24107066\frac{2}{59}$	2	2	12
39	$33491675\frac{1}{49}$	2	2	12
40	$47830565\frac{5}{47}$	2	2	12
41	$71400259\frac{1}{179}$	2	2	12
42	115143336	2	2	12
44	238814520	3	1	14
45	$315542890\frac{610}{1271}$	3	1	14
46	$419023338\frac{46}{53}$	3	1	14
47	$561215167\frac{4037}{5461}$	3	1	14
48	$761656254\frac{6}{11}$	3	1	14
49	$1054475186\frac{8}{17}$	3	1	14
50	$1504942258\frac{16}{23}$	3	1	14
51	$2255135531\frac{583}{1363}$	3	1	14
55	$9437927703\frac{133}{229}$	3	2	16
56	$12438770728\frac{8}{43}$	3	2	16
57	$16538544622\frac{31216}{39785}$	3	2	16

Table 1: (continued)

m	w_m	K	Form	Degree
58	$22282754713\frac{1}{23}$	3	2	16
59	$30616778153\frac{137891}{168533}$	3	2	16
60	43329012480	3	2	16
61	$64250386884\frac{27108}{41039}$	3	2	16
66	$350194168928\frac{6406}{8155}$	4	1	18
67	$461936954642\frac{6152}{63809}$	4	1	18
68	$616741349591\frac{433}{1237}$	4	1	18
69	$838108246881\frac{6081}{9829}$	4	1	18
70	1169132164200	4	1	18
71	$1698060388955\frac{28265}{298283}$	4	1	18
77	$12411939766938\frac{246}{511}$	4	2	20
78	$16405689281448\frac{24}{37}$	4	2	20
79	$22010329107332\frac{140248}{248905}$	4	2	20
80	$30177990957237\frac{17}{19}$	4	2	20
81	$42744922122472\frac{42898}{53507}$	4	2	20
82	$63758049542988\frac{64}{65}$	4	2	20
89	$561945080967167\frac{37795}{1048459}$	5	1	22
90	$757112348026609\frac{2267}{6637}$	5	1	22
91	$1045988435188294\frac{5400226}{9600141}$	5	1	22
92	$1501230731871410\frac{2}{257}$	5	1	22
112	$816451378740698787\frac{5}{9}$	6	1	26
113	$1136254535300176728\frac{151306698}{223651759}$	6	1	26
114	$1136254535300176728\frac{151306698}{223651759}$	6	1	26

In the table 2 the values w_m^A are given for the case when an extremal polynomial in the problem (6) (for $s = 1/2$) is a polynomial of the form 3 or 4. These values can be used as new estimates of the largest power of spherical 1/2-codes in $\mathbb{R}^m$ which are unimprovable by the Delsarte method.

Table 2

m	w_m^A	K	Form	Degree
3	$12.8340776752\ldots$	1	3	10
33	$5203280.6707141049\ldots$	2	4	16
43	$170133239.5931416562\ldots$	3	3	18
52	$3506589575.3297508814\ldots$	3	4	20
53	$4906979442.0645648056\ldots$	3	4	20
54	$6965642842.5492071202\ldots$	3	4	20
62	$95994610190.3413097554\ldots$	4	3	22
63	$131582414832.0133343262\ldots$	4	3	22
64	$182480513596.8192404599\ldots$	4	3	22
65	$257327059360.7694099942\ldots$	4	3	22
72	$2512477187944.7980147749\ldots$	4	4	24
73	$3382770986274.7717090200\ldots$	4	4	24
74	$4595841393803.8776672113\ldots$	4	4	24
75	$6325468915069.0951433926\ldots$	4	4	24
76	$8869434642969.6959845223\ldots$	4	4	24
83	$84893140132749.4433303610\ldots$	5	3	26
84	$113336757486228.6751081110\ldots$	5	3	26
85	$152780346952246.4470059715\ldots$	5	3	26
86	$208816888813525.1041282031\ldots$	5	3	26
87	$291111233760547.9215716098\ldots$	5	3	26
88	$417810824838712.4573733198\ldots$	5	3	26
93	$2101339201083448.8581013596\ldots$	5	4	28
94	$2765414429020562.7271354169\ldots$	5	4	28
95	$3664279182837636.4124486346\ldots$	5	4	28
96	$4903834748673690.9286701217\ldots$	5	4	28
97	$6656234865994295.8866755890\ldots$	5	4	28
98	$9219414667236866.7539438955\ldots$	5	4	28
99	$13154452413669110.8651309899\ldots$	5	4	28
104	$67128787164683544.4523776297\ldots$	6	3	30
105	$87735771631740197.9352556672\ldots$	6	3	30
106	$115452705604234426.2027477576\ldots$	6	3	30
107	$153434175553796665.9933019243\ldots$	6	3	26
108	$206786189887861292.3744524831\ldots$	6	3	30
109	$284301510798701394.0821487141\ldots$	6	3	30
110	$402436933410128596.4722278511\ldots$	6	3	30

Table 2: (continued)

m	w_m	K	Form	Degree
111	595836902535559024.3188262687...	6	3	30
115	2101589801997124293.4926488050...	6	4	32
116	2729686255505205518.5715664555...	6	4	32
117	3568896545606734776.3884783889...	6	4	32
118	4710751164134714024.1028237957...	6	4	32
119	6302155682133310144.5379747181...	6	4	32
120	8593437376247698936.3526858911...	6	4	32
121	12046660361010217345.2333308572...	6	4	32
122	17615670890375383613.3569976086...	6	4	32
125	50364867085052138405.0507562145...	7	3	34
126	64793425556670001358.5275796370...	7	3	34
127	83682707580452106791.2097623968...	7	3	34
128	108746230244493146881.2976319596...	7	3	34
129	142584503530083915887.6303725657...	7	3	34
130	189322485836170786553.6385557962...	7	3	34
131	255885036015648050531.0181807358...	7	3	34
132	354803789323254565081.7957039872...	7	3	34
133	511236181565006961686.6864799904...	7	3	34
134	784028296517640221309.3273031865...	7	3	34
136	1541633134566897630439.9212931033...	7	4	36
137	1973936094367255380618.5869006801...	7	4	36
138	2536282160116347359248.0508958835...	7	4	36
139	3277120814346441078193.1926393845...	7	4	36
140	4269133305200474922172.0302255769...	7	4	36
141	5625976046648898147250.0406685544...	7	4	36
142	7535031448888363741834.1039332556...	7	4	36
143	10327388458447120861753.9448151515...	7	4	36
144	14645940302969237174125.6523056427...	7	4	36
145	21920980271484567377147.0958895900...	7	4	36
147	46712811159099187620845.3387817426...	8	3	38
148	59565056612990122680415.8493932533...	8	3	38
149	76177853174918430594445.5483347921...	8	3	38
150	97906064392590213580625.1938449228...	8	3	38
151	126755987190626475051892.3954775655...	8	3	38
152	165815400461133042447385.7661426883...	8	3	38

Table 2: (continued)

m	w_m	K	Form	Degree
153	220073534234554737523908.4115945786...	8	3	38
154	298116449057161906517212.6325878034...	8	3	38
155	416023538135200368061705.0307527308...	8	3	38
156	607718608181858421565493.8445181846...	8	3	38
161	2904804593623115788824211.9075380435...	8	4	40

5 The proof for the case $\mathbf{m = 43}$.

Let $m = 43$, then $\alpha = 20$ and the polynomials $R_k = R_k^{20,20}$ are the ultraspherical polynomials orthogonal on $[-1, 1]$ with weight $(1 - t^2)^{20}$ and normalized by the condition $R_k(1) = 1$. In this section we evaluate the quantity

$$w_{43}^A = w_{43}^A(1/2) = \inf\left\{\frac{f(1) + f(-1)}{f_0} : f \in \mathcal{F}_{43}^A\right\}, \quad \mathcal{F}_{43}^A = \mathcal{F}_{43}^A(1/2). \quad (19)$$

According to the table of the results, the extremal function is an even eighteenth-degree polynomial of the form 3 with $K = 3$; its fourteenth and sixteenth coefficients in representation (3) with respect to the ultraspherical polynomials are zero. Thus, the extremal polynomial has the form

$$f^*(t) = \left(t^2 - \frac{1}{4}\right) \cdot \prod_{i=1}^{3}(t^2 - a_i^2)^2 \cdot (t^4 + q\,t^2 + r), \quad f_{14}^* = f_{16}^* = 0. \quad (20)$$

We are to choose parameters of the polynomial, i.e., the zeros $\{a_i\}$ and the coefficients q, r; in addition, the following two conditions must be satisfied:

(c1) the (double) zeros a_i, $1 \le i \le 3$, lie on $(0, 1/2)$;

(c2) the polynomial $t^4 + q\,t^2 + r$ is nonnegative on $[0, 1/2]$ (and, in reality, it will be positive on all the axis).

Simultaneously with the function (20) we are to construct the quadrature formula

$$f_0 = \frac{1}{\vartheta(\alpha)} \int_{-1}^{1} f(t)(1 - t^2)^{\frac{m-3}{2}}\,dt = L(f) - \gamma_{14} f_{14} - \gamma_{16} f_{16}, \quad (21)$$

$$L(f) = \sum_{i=1}^{3} A_i\, f(a_i) + A_0\, f\left(\frac{1}{2}\right) + A_4\, f\,(1)\,, \quad (22)$$

with the following properties:

(p1) the formula is sharp on the set $\mathcal{P}_{18}^{A}$ of even algebraic polynomials of degree up to 18;

(p2) $L(R_{2k}) \geq 0,\ k \geq 0$; in addition (as a consequence of the preceding condition), $L(R_{2k}) = 0,\ 1 \leq k \leq 9,\ k \neq 7, 8$;

(p3) the coefficients $A_i,\ 0 \leq i \leq 4$, of formula (22) are nonnegative.

The number of all the parameters here is twelve. Namely, there are five coefficients A_i, three unknowns a_i, two coefficients of the polynomial $t^4 + q\,t^2 + r$, and two coefficients γ_{14}, γ_{16}. In order to find these parameters, we construct the set Σ_{12} of twelve (nonlinear) equations. Ten equations are given by the condition that the quadrature formula (21) is sharp for even algebraic polynomials of the eighteenth degree, i.e., the condition (p 1); and two equations are given by the conditions that the coefficients $f_{14}^*,\ f_{16}^*$ of representation of the unknown polynomial f^* by the system $\{R_{2k}\}$ are equal to zero.

In order to simplify Σ_{12}, we use the following considerations. We use a basis of polynomials in the space $\mathcal{P}_{18}^{A}$ such that after substitution of these polynomials in the quadrature formula (21) we get the simplest equations. In addition, we replace the squared zeros $a_i,\ 1 \leq i \leq 3$, of the polynomial (20) by their symmetric functions $U_i,\ 1 \leq i \leq 3$. To be exact, we find the polynomial $P_3(t) = \prod_{i=1}^{3}(t^2 - a_i^2)$, whose square is in the expansion (20), in the form

$$P_3(t) = t^6 - U_2 t^4 + U_1 t^2 - U_0.$$

The system of equations Σ_{12} constructed in this way has several solutions, and only one of them satisfies conditions (c1), (c2), (p2), (p3).

One can find the construction of the system of equations and its solution in the proof of the following theorem and Lemma 5.1. We solved the system with the help of the Maple package of analytic computations. We do not give these constructions and computations. We give a result of the computations, i.e., we give the concrete function (20) and the quadrature formula (21) + (22) and prove that they solve the problem.

To state the results of this part of the work, we need some definitions and notation. Let H be the following polynomial of the third degree:

$$H(z) = z^3 - \frac{1835489}{2079100}\,z^2 + \frac{590059779}{1287046064}\,z - \frac{106321508304907}{2129617205027920}. \tag{23}$$

The polynomial has one real and two complex zeros:

$$\begin{aligned} z_1 &= 0.1411134854416294\ldots, \\ z_{2,3} &= 0.3708575711650012\ldots \pm i \cdot 0.4650367196476447\ldots . \end{aligned}$$

Let us agree to denote the zero z_1 of the polynomial H by ξ, so that

$$\xi = z_1 = 0.1411134854416294\ldots\,. \tag{24}$$

Let

$$\begin{aligned} q &= 2\,\xi - \frac{179}{100} = -1.5077730291167411\ldots, \\ r &= 3\,\xi^2 - \frac{3914589}{1039550}\,\xi + \frac{40170654239}{32176151600} = 0.7768145246622512\ldots, \end{aligned}$$

$$\begin{aligned} \zeta_0 &= \frac{779}{1018759}\,\xi - \frac{5570}{53994227}\,, \\ \zeta_1 &= \frac{1930}{20791}\,\xi - \frac{10605}{1101923}\,, \\ \zeta_2 &= \xi\,. \end{aligned} \tag{25}$$

The numbers $q, r, \zeta_0, \zeta_1, \zeta_2$ define the polynomial of the eighteenth degree

$$f^*(t) = \left(t^2 - \frac{1}{4}\right)\left(t^6 - \zeta_2\,t^4 + \zeta_1\,t^2 - \zeta_0\right)^2\left(t^4 + q\,t^2 + r\right). \tag{26}$$

We denote by f_k^* its Fourier coefficients in the expansion with respect to the polynomials $\{R_k\}$. The polynomial f^* is an even polynomial. Therefore its coefficients with odd indexes are equal to zero. Hence the expansion has the following form

$$f^*(t) = \sum_{k=0}^{18} f_{2k}^* R_{2k}(t) \tag{27}$$

Finally, let $\{a_i\}_{i=1}^3$ be positive zeros of the polynomial

$$g(z) = z^6 - \zeta_2\,z^4 + \zeta_1\,z^2 - \zeta_0; \tag{28}$$

they all belong to $(0, 1/2)$ and have the following approximate values:

$$\begin{aligned} a_1 &= 0.0380726602850886\ldots\,, \\ a_2 &= 0.1725867591939439\ldots\,, \\ a_3 &= 0.3314781569445825\ldots\,. \end{aligned} \tag{29}$$

By (26), $\pm a_i,\ 1 \le i \le 3$, are the double zeros of the function f^*.

The following statement is the main result of this section.

Theorem 5.1 *The polynomial f^*, defined by (23) − (26), belongs to the set $\mathcal{F}_{43}^A$ and is the unique (to within a positive constant factor) extremal function (solution) of the problem (19). Moreover, we have*

$$w_{43}^A = \frac{2f^*(1)}{f_0^*} = 170133239.5931416562399728\ldots \ . \tag{30}$$

The proof of Theorem 5.1 requires some preliminary statements.

For the ultraspherical polynomials $R_n = R_n^{\alpha,\alpha}$, $\alpha = (m-3)/2$, for any $m \geq 2$, in particular, for $m = 43$, the following recurrence relation holds (see, for instance, [6, p.64, formula (4.5)]):

$$R_{n+1}(t) = \frac{(2n+m-2)\,t\,R_n(t) - n\,R_{n-1}(t)}{n+m-2}, \quad n \geq 1,$$

$$R_0(t) = 1, \quad R_1(t) = t.$$

This relation is usable for evaluation of the coefficients of the polynomials R_n, $n \geq 1$, in their expansion by degrees of t. In the sequel on several occasions we represent a polynomial $f(t) = \sum_{k=0}^{\nu} c_k(f) t^k$ of degree ν by the system $\{R_k\}$:

$$f(t) = \sum_{k=0}^{\nu} f_k R_k(t). \tag{31}$$

We do this by the following known scheme. Suppose $c_\nu(R_\nu)$ is the leading coefficient of the polynomial R_ν. Then for the expansion (31) we have $f_\nu = c_\nu(f)/c_\nu(R_\nu)$. The degree of the polynomial $f^1 = f - f_\nu\, R_\nu$ is equal to $\nu - 1$. Repeating this process for the polynomial f^1, we get the coefficient $f_{\nu-1}$ and so on.

With the help of the polynomial (28) we define the polynomials

$$g^i(t) = (t^2 - 1/4)(t^2 - 1)\, g(t)/(t^2 - a_i^2) \quad i = 1,\ 2,\ 3. \tag{32}$$

In what follows we use the notation

$$g^i(t) = \sum_{j=0}^{9} g_{2j}^i\, R_{2j}(t) \tag{33}$$

for coefficients in representations of these polynomials by the system of ultraspherical polynomials. Let us introduce the quantities

$$\lambda(a_i) = g_0^i / g^i(a_i), \quad i = 1,\ 2,\ 3. \tag{34}$$

Their numerical values are the following:

$$\begin{aligned}\lambda(a_1) &= 0.4835866972149467\dots\ ,\\ \lambda(a_2) &= 0.4319241073046564\dots\ ,\\ \lambda(a_3) &= 0.0815919008616610\dots\ .\end{aligned} \tag{35}$$

Let also

$$\lambda(1) = -\frac{1}{23009085}\left(\frac{214703\,\xi - 24075}{24221\,\xi - 26423}\right) = 0.00000001175549\dots, \tag{36}$$

$$\lambda\left(\frac{1}{2}\right) = \frac{928514048}{90945}\left(\frac{53\,\xi - 15}{138423916\,\xi - 46036387}\right) = 0.002897282863\dots. \tag{37}$$

The following statement contains the quadrature formula on the set of functions (11) for $m = 43$. The proof of Theorem 5.1 is based on this formula.

Lemma 5.1 *For functions $f(t) = \sum_{j=0}^{\infty} f_{2j} R_{2j}(t) \in \Phi_{43}^{A}$ we have the following quadrature formula:*

$$f_0 = \frac{1412926920405}{549755813888}\int_{-1}^{1} f(t)(1-t^2)^{\frac{43-3}{2}}\,dt = L(f) - \sum_{\nu\ge 1} L(R_{2\nu}) f_{2\nu}; \tag{38}$$

here L is the functional

$$L(f) = \lambda(1) f(1) + \lambda\left(\frac{1}{2}\right) f\left(\frac{1}{2}\right) + \sum_{i=1}^{3} \lambda(a_i) f(a_i) \tag{39}$$

with coefficients defined by (34)—(37)*. The functional L has the following properties:*

$$L(1) = 1; \tag{40}$$

$$L(R_{2\nu}) = 0, \quad \nu = 1,\ 2, \dots,\ 5,\ 6,\ 9; \tag{41}$$

$$L(R_{2\nu}) > 0, \quad \nu = 7,\ 8,\ \nu \ge 10. \tag{42}$$

P r o o f. In the proof of the lemma we use the ideas similar to those applied in the proof of Lemma 4.2 from [12] in the investigations of the quantity $w_4(1/2)$.

We first find what conditions on the real nodes

$$0 < A_1 < A_2 < A_3 < \frac{1}{2} \tag{43}$$

and the coefficients $\lambda_1,\ \lambda_{1/2},\ \lambda_{A_1},\ \lambda_{A_2},\ \lambda_{A_3},\ \gamma_{14},\ \gamma_{16}$ must hold if the quadrature formula

$$f_0 = \frac{1412926920405}{549755813888}\int_{-1}^{1} f(t)\cdot(1-t^2)^{\frac{43-3}{2}}\,dt = \mathcal{L}(f) - \gamma_{14}\,f_{14} - \gamma_{16}\,f_{16}, \quad (44)$$

$$\mathcal{L}(f) = \lambda_1 f(1) + \lambda_{1/2} f\left(\frac{1}{2}\right) + \sum_{i=1}^{3} \lambda_{A_i} f(A_i), \quad (45)$$

is valid on the set of all even polynomials $f(t) = \sum_{k=0}^{9} f_{2k} R_{2k}(t)$ of degree up to 18. Next we add several conditions that together with the previous ones become sufficient for construction of the required quadrature formula on the set Φ_{43}^{A}.

Let

$$\chi(t) = \prod_{i=1}^{3}(t^2 - A_i^2) = t^6 - U_2\,t^4 + U_1\,t^2 - U_0. \quad (46)$$

We consider the following polynomial of the tenth degree:

$$\sigma(t) = (t^2 - 1)\left(t^2 - \frac{1}{4}\right)\chi(t). \quad (47)$$

With the help of the polynomial, we define some more algebraic polynomials (of degree up to 18). Let $\varphi^1 = \sigma$ and

$$\varphi^1(t) = \sum_{k=0}^{9} \varphi_{2k}^1 R_{2k}(t)$$

be the expansion of the polynomial by the system $\{R_k\}_{k=0}^{\infty}$. In fact, the degree of the polynomial is 10 and therefore, $\varphi_{14}^1 = \varphi_{16}^1 = 0$. Our interest is the coefficient of R_0 in the expansion. For this coefficient we have

$$\varphi_0^1 = -\frac{29}{141470}\,U_2 + \frac{49}{12126}\,U_1 - \frac{287}{1290}\,U_0 + \frac{23}{1442994}.$$

Similarly, for the polynomial

$$\varphi^2(t) = t^2\sigma$$

we have

$$\varphi_0^2 \quad = \quad -\frac{23}{1442994}\,U_2 + \frac{29}{141470}\,U_1 - \frac{49}{12126}\,U_0 + \frac{1}{642678}.$$

Substituting the polynomials φ^1, φ^2 in (44), we get the first necessary condition of existence of the formula:

$$\varphi^1_0 = 0, \quad \varphi^2_0 = 0.$$

The last relations give the following system of two linear equations with three unknowns:

$$-\frac{29}{141470}\,U_2 + \frac{49}{12126}\,U_1 - \frac{287}{1290}\,U_0 + \frac{23}{1442994} = 0,$$
$$-\frac{23}{1442994}\,U_2 + \frac{29}{141470}\,U_1 - \frac{49}{12126}\,U_0 + \frac{1}{642678} = 0;$$

this is equivalent to

$$\begin{aligned} U_0 &= \frac{779}{1018759}\,U_2 - \frac{5570}{53994227}, \\ U_1 &= \frac{1930}{20791}\,U_2 - \frac{10605}{1101923}. \end{aligned} \tag{48}$$

We consider the polynomial $\varphi^3(t) = t^4 \cdot \sigma(t)$; it is of the fourteenth degree. For the coefficients in the expansion of the polynomial with respect to the system of the ultraspherical polynomials, we have

$$\varphi^3_0 = -\frac{1}{642678}\,U_2 + \frac{23}{1442994}\,U_1 - \frac{29}{141470}\,U_0 + \frac{3}{18209210},$$
$$\varphi^3_{14} = \frac{847872}{4581527}, \quad \varphi^3_{16} = 0.$$

Substituting the polynomial φ^3 in (44), we get

$$\varphi^3_0 = -\gamma_{14}\,\varphi^3_{14}; \tag{49}$$

this is equivalent to

$$\begin{aligned} \gamma_{14} &= \frac{4581527}{544908681216}\,U_2 - \frac{4581527}{53194530816}\,U_1 + \\ &+ \frac{132864283}{119948451840}\,U_0 - \frac{4581527}{5146359767040}. \end{aligned} \tag{50}$$

Similarly, using the polynomial of the sixteenth degree $\varphi^4(t) = t^6 \cdot \sigma(t)$ and the expression (50) for γ_{14}, we find the coefficient γ_{16} :

$$\begin{aligned} \gamma_{16} &= \frac{325288417}{24323459973120}\,{U_2}^2 - \frac{7481633591}{54613051637760}\,U_1U_2 + \\ &+ \frac{9433364093}{5354220748800}\,U_0U_2 - \frac{543272890133}{11577966947205120}\,U_1 - \\ &- \frac{19925060923}{31207458078720}\,U_0 - \frac{114538175}{19458767978496}\,U_2 + \frac{39304679}{78761679912960}. \end{aligned} \tag{51}$$

Let us introduce the polynomials

$$\begin{aligned} h_i(t) &= \sigma(t)/(t^2-(a_i)^2), \quad 1 \le i \le 3, \\ h_4(t) &= \left(t^2-\frac{1}{4}\right)\chi(t) = \sigma(t)/(t^2-1), \\ h_5(t) &= (t^2-1)\chi(t) = \sigma(t)/(t^2-1/4). \end{aligned}$$

Substituting the polynomials h_4, h_5 in (44), we get

$$\begin{aligned} \lambda_1 &= -\frac{1}{636615}\,\frac{189\,U_2-3619\,U_1+192465\,U_0-15}{U_2-U_1+U_0-1}, \\ \lambda_{1/2} &= \frac{512}{636615}\,\frac{147\,U_2-2303\,U_1+103635\,U_0-15}{4\,U_2-16\,U_1+64\,U_0-1}. \end{aligned}$$

Substituting $U_0,\ U_1$ from (48) in formulas for $\lambda_1,\ \lambda_{1/2}$, we get

$$\lambda_1 = -\frac{1}{23009085}\,\frac{214703\,U_2-24075}{24221\,u_2-26423}, \tag{52}$$

$$\lambda_{1/2} = \frac{928514048}{90945}\,\frac{53\,U_2-15}{138423916\,U_2-46036387}. \tag{53}$$

In the same way, with the help of the polynomials $h_i,\ 1 \le i \le 3$, we find

$$\lambda_{A_i} = h_0^i/h^i(A_i),\ i=1,\ 2,\ 3. \tag{54}$$

We do not give explicit formulas for the coefficients λ_{A_i}, because the expressions occupy too much room.

Substituting the polynomials $\varphi^5(t) = t^8\sigma(t)$ in (44) and considering the formulas obtained above for $\gamma_{14}, \gamma_{16}, U_0, U_1$, we get the following equation:

$$F(U_2) = 0, \tag{55}$$

where

$$F(U_2) = {U_2}^3 - \frac{1835489}{2079100}\,{U_2}^2 + \frac{590059779}{1287046064}\,U_2 - \frac{106321508304907}{2129617205027920}. \tag{56}$$

The polynomial $F(U_2)$ coincides with the polynomial H. Thus, the conditions $H(U_2)=0$ and (48), and (50)—(54) are the necessary conditions for the existence of the quadrature formula (44).

As mentioned above, the polynomial H has one real and two complex zeros; in (24) we denoted its first real zero by ξ. From now on we suppose that

$$U_2 = \zeta_2 = \xi,\ U_1 = \zeta_1,\ U_0 = \zeta_0,$$

where ζ_0, ζ_1, ζ_2 are defined by (25). Under these assumptions, the polynomial χ defined by (46) coincides with the polynomial (28). Consequently,

$$A_i = a_i, \quad i = 1,\ 2,\ 3. \tag{57}$$

This implies that the coefficients $\lambda_1, \lambda_{1/2}, \lambda_{A_1}, \lambda_{A_2}, \lambda_{A_3}$ of the functional $\mathcal{L}$ (see (45)) coincide with the coefficients (34) — (37) of the functional L defined by (39). Therefore, (44) takes the form

$$f_0 = \frac{1412926920405}{549755813888} \int_{-1}^{1} f(t)(1-t^2)^{\frac{43-3}{2}} dt = L(f) - \gamma_{14} f_{14} - \gamma_{16} f_{16}. \tag{58}$$

Now we can assert that the formula is valid for the polynomials $h^1,\ h^2,\ \ldots,\ h^5,\ \varphi^1,\ \varphi^2,\ \ldots,\ \varphi^5$. These ten polynomials form the basis in the set $\mathcal{P}_{18}^A$ of even polynomials of degree up to eighteen. Therefore, the quadrature formula (58) holds for any polynomial from $\mathcal{P}_{18}^A$. Substituting the polynomials $R_{14},\ R_{16}$ in the formula, we obtain

$$\gamma_{14} = L(R_{14}), \quad \gamma_{16} = L(R_{16}). \tag{59}$$

Hence, (58) can be written in the form

$$f_0 = \frac{1412926920405}{549755813888} \int_{-1}^{1} f(t)(1-t^2)^{\frac{43-3}{2}} dt = L(f) - \sum_{\nu=7}^{8} L(R_{2\nu}) f_{2\nu}, \tag{60}$$

where

$$L(R_\nu) = \lambda(1) + \lambda\left(\frac{1}{2}\right) R_\nu\left(\frac{1}{2}\right) + \sum_{i=1}^{3} \lambda(a_i) R_\nu(a_i), \tag{61}$$

$a_1,\ a_2,\ a_3$ are positive zeros of the polynomial given by (28).

Hence, in particular, $L(R_{2\nu}) = 0$ for $\nu = 1,\ 2, \ldots,\ 5,\ 6,\ 9,\ \nu \neq 7,\ 8$, i.e., the property (41) holds. Substituting the polynomial $f(t) \equiv 1$ in (60), we also get (40).

Obviously, (60) can be extended to all the class of functions Φ_{43}^A, if we write this formula in the form (38).

In order to complete the proof, it remains to verify inequalities (42). To validate these inequalities, we use the following two properties of the ultraspherical polynomials $\{R_k\}$:

(1) $R_\nu(1) = 1,\ R_\nu(-1) = (-1)^\nu,\ \nu \geq 0$,

(2) on $(-1, 1)$, the polynomials R_ν converge pointwise to zero as $\nu \to \infty$, to be exact, the estimates (62) hold for the polynomials.

For any $\nu \geq 0$ we have

$$\begin{aligned} L\left(R_\nu\right) &\geq \lambda(1) + \left(\lambda\left(\frac{1}{2}\right) + \sum_{i=1}^{3} \lambda(a_i)\right) \min\left\{R_\nu\left(\frac{1}{2}\right), R_\nu(a_1), \ldots, R_\nu(a_3)\right\} = \\ &= \lambda(1) + (1-\lambda(1)) \min\left\{R_\nu\left(\frac{1}{2}\right), R_\nu(a_1), R_\nu(a_2), R_\nu(a_3)\right\} \geq \\ &\geq \lambda(1) - (1-\lambda(1)) \max\left\{\left|R_\nu\left(\frac{1}{2}\right)\right|, \left|R_\nu(a_1)\right|, \left|R_\nu(a_2)\right|, \left|R_\nu(a_3)\right|\right\}. \end{aligned}$$

Let us use the following efficient estimates for the ultraspherical polynomials

$$|R_n^{\alpha,\alpha}(t)| \leq \frac{A(n,m)}{(1-t^2)^\gamma}, \quad \alpha = (m-3)/2, \quad -1 < t < 1, \tag{62}$$

$$n \geq \max\{3, m-4\}, \;\; m \geq 4,$$

$$A(n,m) = \Gamma\left(\frac{m-1}{2}\right) \frac{\sqrt{2}(2+\sqrt{2})^{m-4}}{(n+1)^{\frac{m-2}{2}}}, \quad \gamma = \frac{m-2}{4}. \tag{63}$$

Estimates of the Jacobi polynomials and, particularly, of the ultraspherical polynomials has a rich history; the estimate (62) is contained in ([16], Lemma 2.1).

By the assumption of the lemma, the estimate (62) can be used for $n \geq 39$. Let us recall that $a_i \in (0, 1/2)$. Therefore, for any $\nu \geq 1$ we have

$$\begin{aligned} \max &\left\{\left|R_\nu\left(\frac{1}{2}\right)\right|, \left|R_\nu(a_1)\right|, \left|R_\nu(a_2)\right|, \left|R_\nu(a_3)\right|\right\} \leq \\ \leq &\frac{\Gamma(21)\sqrt{2}(2+\sqrt{2})^{39}}{(1-1/4)^{\frac{41}{4}} \cdot (\nu+1)^{\frac{41}{2}}} \leq \frac{10^{42}}{(\nu+1)^{\frac{41}{2}}}. \end{aligned}$$

It follows easily that $L\left(R_\nu\right) > 0$ for $\nu \geq 2000$. For other ν the conditions (42) can be verified by direct computation with the use of the Maple program. This completes the proof of Lemma 5.1.

Lemma 5.2 *The function f^* defined by (23)–(26) belongs to the set $\mathcal{F}_{43}^A$.*

P r o o f. The polynomial $\pi_2(t) = t^4 + q\,t^2 + r$ in the right-hand side of (26) is positive on $[-1, 1]$. Therefore,

$$f^*(t) \leq 0, \quad -\frac{1}{2} \leq t \leq \frac{1}{2}. \tag{64}$$

It remains to prove that the coefficients f^*_{2k} in the representation (27) of the function (26) are nonnegative; moreover, $f^*_0 > 0$. We evaluate the coefficients of the representation

$$f^*(t) = \left(t^2 - \frac{1}{4}\right)\left(t^6 - \zeta_2\, t^4 + \zeta_1\, t^2 - \zeta_0\right)^2 (t^4 + q\, t^2 + r)$$

with respect to the system of ultraspherical polynomials:

$$\begin{aligned}
f^*_{18} &= \frac{439025664}{6248961695}, \qquad (65)\\
f^*_{16} &= f^*_{14} = 0\,,\\
f^*_{12} &= 0.04176362115799827203492 65\ldots\,,\\
f^*_{10} &= 0.029222891409500233312213 31\ldots\,,\\
f^*_{8} &= 0.0078138941286457485943531\ldots\,,\\
f^*_{6} &= 0.0009446460463425510583249\ldots\,,\\
f^*_{4} &= 0.0000488080874818309645154\ldots\,,\\
f^*_{2} &= 0.0000008242017491816358353\ldots\,,\\
f^*_{0} &= 0.0000000017639878907626135\ldots\,,
\end{aligned}$$

Thus, the representation (3) of the function f^* has nonnegative coefficients and $f^*_0 > 0$. This completes the proof of Lemma 5.2.

The following lemma will allow us to show that the function f^* is the unique (to within a positive constant factor) extremal function of the problem (19).

Lemma 5.3 *Suppose that f is an even polynomial of degree up to eighteen with the following properties:*

(1) *the representation*

$$f(t) = h(t)e(t) \qquad (66)$$

holds, where

$$h(t) = \left(t^2 - \frac{1}{4}\right)\prod_{i=1}^{3}(t^2 - a_i^2)^2,$$

and e is some even polynomial of degree up to four,

(2) *in the representation*

$$f(t) = \sum_{\nu=0}^{18} f_\nu R_\nu(t) \qquad (67)$$

of the polynomial f by the polynomials R_k, the fourteenth and sixteenth coefficients are equal to zero:

$$f_{14} = 0, \quad f_{16} = 0. \tag{68}$$

Then the polynomial f coincides to within a constant factor with the function (polynomial) f^*, i.e., $f = cf^*$, where $c = \mathrm{const}$; in addition, if $f \in \mathcal{F}_{43}^A$, then $c > 0$.

P r o o f. The proof of the lemma will be divided into several steps.

(1) We first suppose that the polynomial e in the representation (66) has the form

$$e(t) = t^4 + e_1 t^2 + e_0. \tag{69}$$

Let us write out the explicit expressions for the coefficients f_{14} and f_{16} in the representation (67) of the polynomial f. It is easily shown that

$$f(t) = f_{18}R_{18}(t) + f_{16}R_{16}(t) + f_{14}R_{14}(t) + \varphi(t), \tag{70}$$

where

$$f_{18} = \frac{439025664}{6248961695},$$

$$f_{16} = -\frac{75694080}{325288417}\,\xi + \frac{37847040}{325288417}\,e_1 + \frac{338731008}{1626442085},$$

$$\begin{aligned} f_{14} \quad = \quad & -\frac{3372580030464}{6763071477847}\,\xi + \frac{847872}{4581527}\,\xi^2 + \frac{847872}{4581527}\,e_0 + \\ + \quad & \frac{86694912}{325288417}\,e_1 - \frac{1695744}{4581527}\,e_1\xi + \frac{6437476926996480}{26166323547790043}, \end{aligned}$$

and, finally, φ is an even polynomial of the twelfth degree.

Let the function f satisfy the conditions (68), i.e., let the right-hand sides of the last two relations be equal to zero. As a result, we obtain a system of two linear equations with two unknowns e_1 and e_0. The system determinant Δ has the form

$$\Delta = -\frac{523813926526333374720}{12990658443511758809 63}+\frac{391631316889632768 0}{2451067630851275247 1}\,\xi-\frac{2595980574720}{16604338082711}\,\xi^2.$$

It follows easily that by (24), $\Delta = -0.0208885275\ldots \neq 0$. Consequently, the system under consideration, i.e., the system of two conditions (68), has the unique solution e_0, e_1. By (26), the function f^* has the representation (66), (69) and by (65), the conditions (68) hold for this function. Hence, $e_1 = q$

and $e_0 = r$. Thus, $f = f^*$ is a unique function satisfying the requirements (66), (69) and (68).

(2) Suppose now that the polynomial e in the representation (66) of the function f has the form $e(t) = ct^4 + e_1 t^2 + e_0$ and $c \neq 0$. Then the function $f_c = \dfrac{f}{c}$ satisfies all the conditions of the lemma and the second factor in the representation (66) of the function has the form (69). It follows, as we have shown just now, that $\dfrac{f}{c} = f^*$ or, equally, $f = cf^*$. Clearly, if $f \in \mathcal{F}^A_{43}$, then $c > 0$.

(3) Finally, let us show that if the function f satisfies the assumptions of the lemma and the second factor in (66) has the form $e(t) = e_1 t^2 + e_0$, then $f \equiv 0$. Indeed, the function $\overline{f} = f^* + f$ satisfies all the assumptions of the lemma. For this function the formula $\overline{f} = h\overline{e}$ is valid, where $\overline{e}$ is the polynomial of the fourth degree $\overline{e}(t) = t^4 + \overline{e}_1 t^2 + \overline{e}_0$ with the coefficients $\overline{e}_1 = q + e_1,\ \overline{e}_0 = r + e_0$. The polynomial $\overline{e}$ has the form (69). Hence, as we have already proved, $\overline{f} = f^*$ and therefore $f \equiv 0$. This completes the proof of Lemma 5.3.

P r o o f of Theorem 5.1. For the function f^* defined by (26), (27) we have

$$f^*\left(\frac{1}{2}\right) = f^*(a_1) = f^*(a_2) = f^*(a_3) = 0, \quad f^*_{14} = f^*_{16} = 0.$$

Therefore, for this function, inequality (38) turns into equality

$$f^*_0 = \lambda(1) f^*(1).$$

This implies that

$$\frac{f^*(1)}{f^*_0} = \frac{1}{\lambda(1)} = -23009085 \left(\frac{24221\,\xi - 26423}{214703\,\xi - 24075}\right).$$

By Lemma 5.2, the function f^* belongs to the set $\mathcal{F}^A_{43}$. Therefore,

$$w^A_{43} \leq \frac{2}{\lambda(1)} = \frac{2 \cdot f^*(1)}{f^*_0}.$$

Thus, in order to prove (30), it is sufficient to show that

$$w^A_{43} \geq \frac{2}{\lambda(1)}. \tag{71}$$

For any function $f \in \mathcal{F}_{43}^A$ the quadrature formula (38)—(39) is valid. The coefficients $\lambda(1), \lambda\left(\frac{1}{2}\right), \lambda(a_1), \lambda(a_2), \lambda(a_3)$ of the formula are positive, and the coefficients $L(R_{2\nu}),\ \nu \geq 1$, are nonnegative. By properties (3)—(5) of the function $f \in \mathcal{F}_{43}^A$, we have the following estimate:

$$f_0 = L(f) - \sum_{\nu \geq 1} L(R_{2\nu}) f_{2\nu} \leq \lambda(1) f(1). \tag{72}$$

This implies inequality (71). This proves the assertion (30) and simultaneously proves that the function f^* is extremal in the problem (19).

In order to prove the theorem, it remains to check that the function f^* is the unique, to within a positive constant factor, extremal function. The even function

$$f(t) = \sum_{\nu=0}^{\infty} f_{2\nu} R_{2\nu}(t) \in \mathcal{F}_{43}^A \tag{73}$$

is extremal iff inequality (72) turns into equality for this function. By the property (42), it is necessary for this that the coefficients f_ν in the representation (73) of the extremal function f be equal to zero for $\nu \neq 0,\ 1,\ 2,\ \ldots,\ 6,\ 9$. In particular, this means that the function f is a polynomial of degree up to eighteen. Besides, the equality $L(f) = \lambda(1) f(1)$ must hold. This equality is equivalent to the following conditions:

$$f\left(\frac{1}{2}\right) = f(a_1) = f(a_2) = f(a_3) = 0.$$

Thus, the point $1/2$ is at least a zero and the points $a_1,\ a_2,\ a_3$ are double zeros of the polynomial f. Consequently, f has the form

$$f(t) = h(t) e(t), \tag{74}$$

where

$$h(t) = \left(t^2 - \frac{1}{4}\right) \prod_{i=1}^{3} (t^2 - a_i^2)^2,$$

and e is a polynomial of degree up to four. Hence, by Lemma 5.3, $f = cf^*$, where c is a positive constant. This completes the proof of Theorem 5.1.

The author wishes to express his thanks to V.V. Arestov for suggesting the interesting problem, his attention to the work, and help in the preparation of this paper. The author is also grateful to A.G. Babenko for his active interest in the investigation.

References

[1] Delsarte, Ph., Bounds for Unrestricted Codes, by Linear Programming, *Phillips Res. Rep.*, 1972, vol. 27, pp. 272–289.

[2] Delsarte, Ph., An Algebraic Approach to the Association Schemes of Coding Theory, *Phillips Res. Reports Supplements*, 1973, no. 10.

[3] Kabatianskii, G.A. and Levenshtein, V.I., Bounds for Packings on a Sphere and in Space, *Problemy Peredachi Informatsii*, 1978, vol. 14, no. 1, pp. 3–25; Engl. transl., *Problems of Information Transmission*, 1978, vol. 14, no. 1, pp. 1–17.

[4] Odlyzko, A.M. and Sloane, N.J.A., New Bounds on the Number of Unit Spheres that Can Touch a Unit Sphere in n Dimensions, *J. of Combinatorial Theory. Ser. A* 26, 1979, pp. 210–214.

[5] Levenshtein, V.I., On Bounds for Packings in n-dimensional Euclidean Space, *Dokl. Akad. Nauk SSSR*, 1979, vol. 245, pp. 1299–1303; Engl. transl., *Soviet Math. Dokl.*, 1979, vol. 20, pp. 417–421.

[6] Levenshtein, V.I., Bounds for Packings in Metric Spaces and Certain Applications, *Problemy Kibernetiki*, 1983, vol. 40, pp. 44–110.

[7] Levenshtein, V.I., Designs as Maximum Codes in Polynomial Metric Spaces, *Acta Applicandae Math.*, 25 (1992), pp. 1–83.

[8] Sidel'nikov, V.M., On Extremal Polynomials Used to Estimate the Size of Codes, *Problemy Peredachi Informatsii*, 1980, vol. 16, no. 3, pp. 17–30; Engl. transl., *Problems of Information Transmission*, 1980, vol. 16, pp. 174–186.

[9] Andreev, N.N. and Yudin, V.A., An Arithmetical Minimum of a Quadratic Form and Spherical Codes, *Matematicheskoe Prosveschenie* (The third series), 1998, iss. 2, pp. 133–140.

[10] Boyvalenkov, P.G., Non existance of certain symmetrical codes, *Designs, Codes, Cryptography*, 3 (1993), pp. 69–74.

[11] Boyvalenkov, P.G., Danev, D.P. and Bumova, S.P., Upper Bounds on the Minimum Distance of Spherical Codes, *IEEE Transactions on Information Theory*, vol. 42, No. 5 (1996), pp. 1576-1581.

[12] Arestov, V.V. and Babenko, A.G., On Delsarte Scheme of Estimating the Contact Numbers, *Tr. Mat. Inst. Steklova*, 1997, vol. 219, pp. 44–73; Engl. transl., *Proc. Steklov Inst. Math.*, 1997, vol. 219, pp. 36–65.

[13] Arestov, V.V. and Babenko, A.G., Estimates of the Maximal Value of Angular Code Distance for 24 and 25 Points on the Unit Sphere in $\mathbf{R}^4$, *Mat. Zametki*, 2000, vol. 68, no. 4, pp. 483–503; Engl. transl., *Math. Notes*, 2000, vol. 68, no. 4, pp. 419–435.

[14] Boyvalenkov, P.G. and Danev, D.P., On linear programming bounds for codes in polynomial metric spaces (Russian), *Problemy Peredachi Informatsii* 34 (1998), no. 2, 16–31; *translation in Problems Inform. Transmission* 34 (1998), no. 2, 108–120.

[15] Arestov, V.V., Babenko, A.G. and Deikalova, M.V., The Delsarte Problem Connected with Spherical 1/3 code in Euclidean Spaces of Dimensions 4, 5, 6, *Proc. Int. Stechkin School on Theory of Functions, Miass, 1998,* Ekaterinburg: UrO RAN, 1999, pp. 5–37.

[16] Shtrom, D.V., The Delsarte Method in the Problem on the Contact Numbers of Euclidean Spaces of High Dimensions, *Proceedings of the Steklov Institute of Mathematics*, Suppl. 2, 2002, pp. S162–S189. Translated from Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2002, vol. 8, no. 2.

[17] Conway, J.H. and Sloane, N.J.A., *Sphere Packings, Lattices and Groups*, New York: Springer, 1988.

[18] Boyvalenkov P.G., Danev D.P., Mitradjieva M. Linear Programming bounds for codes in infinite projective spaces, *Journal of Geometry*, 66 (1999), no. 1–2, pp. 42–54.

[19] Delsarte, P., Goethalts, J. M. and Seidel, J.J., Spherical codes and designs, *Geom. Dedic.* 6 (1977), pp. 363–388.

[20] Gol'shtein, E.G., *Teoriya dvoistvennosti v matematicheskom programmirovanii i ee prilozheniya* (Theory of Duality in Mathematical Programming and Its Applications), Moscow: Nauka, 1971.